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CYLINDRICAL BENDING OF ANGLE-PLY COMPOSITE PLATES

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Abstract: This paper presents an efficient method of solving the plane strain problem of elasticity for laminated composite plates. The method is based on the new concept of sampling surfaces developed recently by the authors. According to this concept, we introduce inside the *n*th layer I_n not equally spaced SaS parallel to the middle surface of the plate and choose displacements of these surfaces as plate unknowns. This fact gives an opportunity to derive the solutions of plane strain elasticity for thick and thin laminated composite plates with a prescribed accuracy by using a sufficiently large number of SaS, which are located at layer interfaces and Chebyshev polynomial nodes.

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1. Introduction

A conventional way for developing the higher order discrete-layer plate theory accounting for thickness stretching is to utilize either quadratic or cubic series expansions in the transverse coordinate for each layer and to choose as unknowns the generalized displacements of layers [1–3]. Herein, we consider a new method of sampling surfaces (**SaS**) inside the plate body proposed recently by the authors [4, 5]. As SaS denoted by $\Omega^{(n)1}, \Omega^{(n)2},..., \Omega^{(n)I_n}$, we choose outer surfaces and any inner surfaces inside the *n*th layer and introduce displacement vectors $\mathbf{u}^{(n)1}, \mathbf{u}^{(n)2},..., \mathbf{u}^{(n)I_n}$ of these surfaces as fundamental plate unknowns, where I_n is the total number of SaS chosen for each layer ($I_n \geq 3$). The index *n* identifies the belonging of any quantity to the *n*th layer and runs from 1 to *N*, where *N* is the number of layers. Such choice of displacements with the consequent use of Lagrange polynomials of degree $(I_n - 1)$ in the thickness direction for each layer permits the representation of governing equations of the laminated plate theory developed in a very compact form.

However, the above polynomial interpolation implemented for equally located SaS [4, 5] does not work properly with Lagrange polynomials of high degree because Runge's phenomenon [6] can occur, which yields the wild oscillation at the edges of the interval when the user deals with any specific functions. If the number of equally spaced nodes is increased then the oscillations become even larger. Fortunately, the use of Chebyshev polynomial nodes [7] can help to improve significantly the behaviour of Lagrange polynomials of high degree for which the error will go to zero as $I_n \to \infty$. This fact gives an opportunity to derive the exact solutions of plane strain elasticity for thick laminated composite plates with a prescribed accuracy utilizing a sufficiently large number of not equally spaced SaS.

2. Three-dimensional description of laminated plate

Consider a thick laminated plate of the thickness h. Let the midsurface Ω be referred to Cartesian coordinates x_1 and x_2, whereas the transverse coordinate x_3 is oriented along the normal direction to the midsurface. The transverse coordinates of SaS inside the nth layer are defined as:

$$x_3^{(n)1} = x_3^{[n-1]}, \quad x_3^{(n)I_n} = x_3^{[n]};$$

$$x_3^{(n)m_n} = \frac{1}{2}(x_3^{[n-1]} + x_3^{[n]}) - \frac{1}{2} h_n \cos\left(\pi \frac{2m_n - 3}{2(I_n - 2)} \right), \tag{1}$$

where $x_3^{[n-1]}$ and $x_3^{[n]}$ are the transverse coordinates of the bottom and top surfaces $\Omega^{[n-1]}$ and $\Omega^{[n]}$ of the n^{th} layer (Fig. 1) such that $x_3^{[0]} = -h/2$ and $x_3^{[N]} = h/2$; $h_n = x_3^{[n]} - x_3^{[n-1]}$ is the thickness of the n^{th} layer; the index m_n identifies the belonging of any quantity to inner SaS of the n^{th} layer and runs from 2 to $I_n - 1$, whereas the indices i_n, j_n, k_n to be introduced in the next section for describing all SaS of the nth layer run from 1 to I_n.

It is important to note that transverse coordinates of inner SaS (1) coincide with the nodes of Chebyshev polynomials [7]. This fact has a great meaning for a convergence of the SaS method.

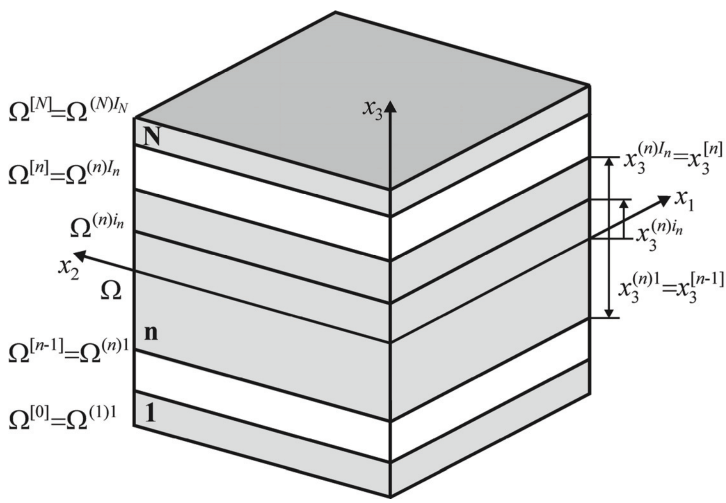

Fig. 1. Geometry of the laminated plate

The strain tensor is given by

$$2\varepsilon_{ij} = u_{i,j} + u_{j,i}, \tag{2}$$

where u_i are the displacements of the plate. Here and in the following developments, Latin tensorial indices i, j, k, ℓ range from 1 to 3, whereas Greek tensorial indices α, β range from 1 to 2.

The strain components of SaS can be written as:

$$2\varepsilon_{\alpha\beta}^{(n)i_n} = 2\varepsilon_{\alpha\beta}(x_3^{(n)i_n}) = u_{\alpha,\beta}^{(n)i_n} + u_{\beta,\alpha}^{(n)i_n};$$

$$2\varepsilon_{\alpha 3}^{(n)i_n} = 2\varepsilon_{\alpha 3}(x_3^{(n)i_n}) = \beta_{\alpha}^{(n)i_n} + u_{3,\alpha}^{(n)i_n};$$

$$2\varepsilon_{33}^{(n)i_n} = 2\varepsilon_{33}(x_3^{(n)i_n}) = \beta_3^{(n)i_n}, \tag{3}$$

where $u_i^{(n)i_n}(x_1, x_2)$ are the displacements of SaS of the n^{th} layer; $\beta_i^{(n)i_n}(x_1, x_2)$ are the derivatives of displacements with respect to coordinate x_3 at SaS, that is,

$$u_i^{(n)i_n} = u_i(x_3^{(n)i_n}); \quad \beta_i^{(n)i_n} = u_{i,3}(x_3^{(n)i_n}). \tag{4}$$

It is convenient to introduce the displacements of bottom and top surfaces of the plate and layer interfaces as follows:

$$u_i^{(1)1} = u_i^{[0]}; \quad u_i^{(N)I_N} = u_i^{[N]};$$

$$u_i^{(m)I_m} = u_i^{(m+1)1} = u_i^{[m]}, \tag{5}$$

where $u_i^{[m]}(x_1, x_2)$ are the displacements of layer interfaces $\Omega^{[m]}$ ($m = 1, 2, ..., N-1$).

3. Displacement and strain distributions in thickness direction

Up to this moment, no assumptions concerning displacement and strain fields have been made. We start now with the first fundamental assumption of the proposed higher order layer-wise plate theory. Let us assume that the displacements are distributed through the thickness of the n^{th} layer as follows:

$$u_i^{(n)} = \sum_{i_n} L^{(n)i_n} u_i^{(n)i_n}; \quad x_3^{[n-1]} \le x_3 \le x_3^{[n]}, \tag{6}$$

where $L^{(n)i_n}(x_3)$ are the Lagrange polynomials of degree $(I_n - 1)$ expressed as

$$L^{(n)i_n} = \prod_{j_n \ne i_n} \frac{x_3 - x_3^{(n)j_n}}{x_3^{(n)i_n} - x_3^{(n)j_n}}. \tag{7}$$

The use of relations (4) and (6) yields

$$\beta_i^{(n)i_n} = \sum_{j_n} M^{(n)j_n}(x_3^{(n)i_n}) u_i^{(n)j_n}, \tag{8}$$

where $M^{(n)j_n} = L_{,3}^{(n)j_n}$ are the derivatives of Lagrange polynomials. The values of these derivatives at SaS of the n^{th} layer are calculated as:

$$M^{(n)j_n}\left(x_3^{(n)i_n}\right)=\frac{1}{x_3^{(n)j_n}-x_3^{(n)i_n}}\prod_{k_n\neq i_n,\,j_n}\frac{x_3^{(n)i_n}-x_3^{(n)k_n}}{x_3^{(n)j_n}-x_3^{(n)k_n}} \quad \text{for} \quad j_n\neq i_n;$$

$$M^{(n)i_n}\left(x_3^{(n)i_n}\right)=-\sum_{j_n\neq i_n}M^{(n)j_n}\left(x_3^{(n)i_n}\right). \tag{9}$$

The latter formula is valid because a useful identity for derivatives of the Lagrange polynomials

$$\sum_{j_n}M^{(n)j_n}=0 \tag{10}$$

holds. Thus, the key functions $\beta_i^{(n)i_n}$ of the proposed higher order layer-wise plate theory are represented according to (8) as a linear combination of displacements of SaS of the nth layer $u_i^{(n)j_n}$.

The following step consists in a choice of the correct approximation of strains through the thickness of the nth layer. It is apparent that the optimal solution of the problem is to choose the strain distribution, which is similar to the displacement distribution (6), that is,

$$\varepsilon_{ij}^{(n)}=\sum_{i_n}L^{(n)i_n}\varepsilon_{ij}^{(n)i_n}; \qquad x_3^{[n-1]}\leq x_3\leq x_3^{[n]}. \tag{11}$$

It is necessary to note that strain-displacement relationships (3) and (11) exactly represent all rigid-body motions of the laminated plate. A proof of this statement can be done following a technique developed in [8–12].

4. Total potential energy of laminated plate

Substituting strains (11) in the total potential energy of a laminated plate and introducing stress resultants

$$H_{ij}^{(n)i_n}=\int_{x_3^{[n-1]}}^{x_3^{[n]}}\sigma_{ij}^{(n)}L^{(n)i_n}dx_3, \tag{12}$$

one obtains

$$\Pi=\iint_{\Omega}\left[\frac{1}{2}\sum_{n}\sum_{i_n}\sum_{i,j}H_{ij}^{(n)i_n}\varepsilon_{ij}^{(n)i_n}-\sum_{i}\left(p_i^{[N]}u_i^{[N]}-p_i^{[0]}u_i^{[0]}\right)\right]dx_1dx_2-W_{\Sigma}, \tag{13}$$

where $p_i^{[0]}$ and $p_i^{[N]}$ are the loads acting on the bottom and top surfaces $\Omega^{[0]}$ and $\Omega^{[N]}$; W_Σ is the work done by external loads applied to the boundary surface Σ.

For simplicity, we restrict ourselves to the case of linear elastic materials. The natural choice for constitutive equations is the generalized Hook's law:

$$\sigma_{ij}^{(n)} = \sum_{k,\ell} C_{ijk\ell}^{(n)} \varepsilon_{k\ell}^{(n)}; \qquad x_3^{[n-1]} \le x_3 \le x_3^{[n]}. \tag{14}$$

Inserting stresses (14) in (12) and taking into account the strain distribution (11), we have

$$H_{ij}^{(n)i_n} = \sum_{j_n} \sum_{k,\ell} D_{ijk\ell}^{(n)i_n j_n} \varepsilon_{k\ell}^{(n)j_n}, \tag{15}$$

where

$$D_{ijk\ell}^{(n)i_n j_n} = C_{ijk\ell}^{(n)} \int_{x_3^{[n-1]}}^{x_3^{[n]}} L^{(n)i_n} L^{(n)j_n} dx_3. \tag{16}$$

5. Exact solution for laminated composite plate

Consider a simply supported laminated plate in cylindrical bending subjected to the sinusoidally distributed transverse load

$$p_3^{[N]} = p_0 \sin\frac{\pi x_1}{a}, \tag{17}$$

where a is the width of the plate.

To satisfy the boundary conditions, we search the analytical solution of the problem as follows:

$$u_1^{(n)i_n} = u_{10}^{(n)i_n} \cos\frac{\pi x_1}{a}; \quad u_2^{(n)i_n} = u_{20}^{(n)i_n} \cos\frac{\pi x_1}{a}; \quad u_3^{(n)i_n} = u_{30}^{(n)i_n} \sin\frac{\pi x_1}{a}. \tag{18}$$

Substituting (17) and (18) into the total potential energy (13) with $W_\Sigma = 0$ and allowing for relations (3), (8) and (15), one finds

$$\Pi = \Pi(u_{i0}^{(n)i_n}). \tag{19}$$

Invoking the principle of the minimum total potential energy, we arrive at the system of linear algebraic equations

$$\frac{\partial \Pi}{\partial u_{i0}^{(n)i_n}} = 0 \tag{20}$$

of order $3\left(\sum_n I_n - N + 1\right)$. The linear system (20) can be easily solved by using a method of Gaussian elimination.

The described algorithm was performed with the Symbolic Math Toolbox, which incorporates symbolic computations into the numeric environment of MatLab. The latter gave the possibility to derive the exact solutions of the plane strain problem of elasticity for angle-ply composite plates with a prescribed accuracy.

Consider a symmetric three-ply plate with a stacking sequence $[30^\circ/-30^\circ/30^\circ]$. The mechanical and geometrical parameters of the simply supported plate are taken to be $h_1 = h_3 = 0{,}25$; $h_2 = 0{,}5$; $E_L = 25E_T$; $G_{LT} = 0{,}5E_T$; $G_{TT} = 0{,}2E_T$; $E_T = 10^6$; $\nu_{LT} = \nu_{TT} = 0{,}25$, where subscripts L and T refer to the fiber and transverse directions of the ply. To compare the results derived with Pagano's exact solution [13], the following dimensionless variables are introduced:

$$U_3 = 100E_T h^3 u_3(a/2, z)/p_0 a^4;$$

$$S_{11} = 10h^2\sigma_{11}(a/2, z)/p_0 a^2; \quad S_{12} = 10h^2\sigma_{12}(0, z)/p_0 a^2;$$

$$S_{\alpha 3} = 10h\sigma_{\alpha 3}(0, z)/p_0 a; \quad S_{33} = \sigma_{33}(a/2, z)/p_0, \quad z = \theta_3/h. \tag{21}$$

Results for a three-ply plate

I_n	$U_3(-0{,}5)$	$U_3(0)$	$U_3(0{,}5)$	$S_{11}(0{,}5)$	$S_{12}(0{,}5)$	$S_{13}(0)$	$S_{23}(0{,}3)$	$S_{33}(0{,}5)$
				***a/h* = 2**				
3	7,9961	8,6849	10,855	14,499	–6,6104	4,0347	–1,0567	1,0289
5	8,2414	8,9154	11,104	15,180	–6,9178	4,6995	–1,0003	1,0011
7	8,2432	8,9178	11,106	15,183	–6,9189	4,6296	–1,0106	1,0000
9	8,2432	8,9178	11,106	15,183	–6,9189	4,6330	–1,0106	1,0000
11	8,2432	8,9178	11,106	15,183	–6,9189	4,6329	–1,0106	1,0000
				***a/h* = 4**				
3	3,1923	3,2531	3,2531	8,3971	–4,0226	4,3664	–1,2610	1,0327
5	3,2305	3,2900	3,4138	8,5205	–4,0811	5,0215	–1,2566	1,0003
7	3,2306	3,2901	3,4138	8,5205	–4,0811	5,0019	–1,2578	1,0000
9	3,2306	3,2901	3,4138	8,5205	–4,0811	5,0022	–1,2578	1,0000
11	3,2306	3,2901	3,4138	8,5205	–4,0811	5,0022	–1,2578	1,0000
				***a/h* = 100**				
3	0,84662	0,84665	0,84662	6,0831	–3,2257	4,3804	–1,6241	1,0393
5	0,84663	0,84666	0,84663	6,0832	–3,2258	4,7758	–1,6199	1,0000
7	0,84663	0,84666	0,84663	6,0832	–3,2258	4,7758	–1,6199	1,0000

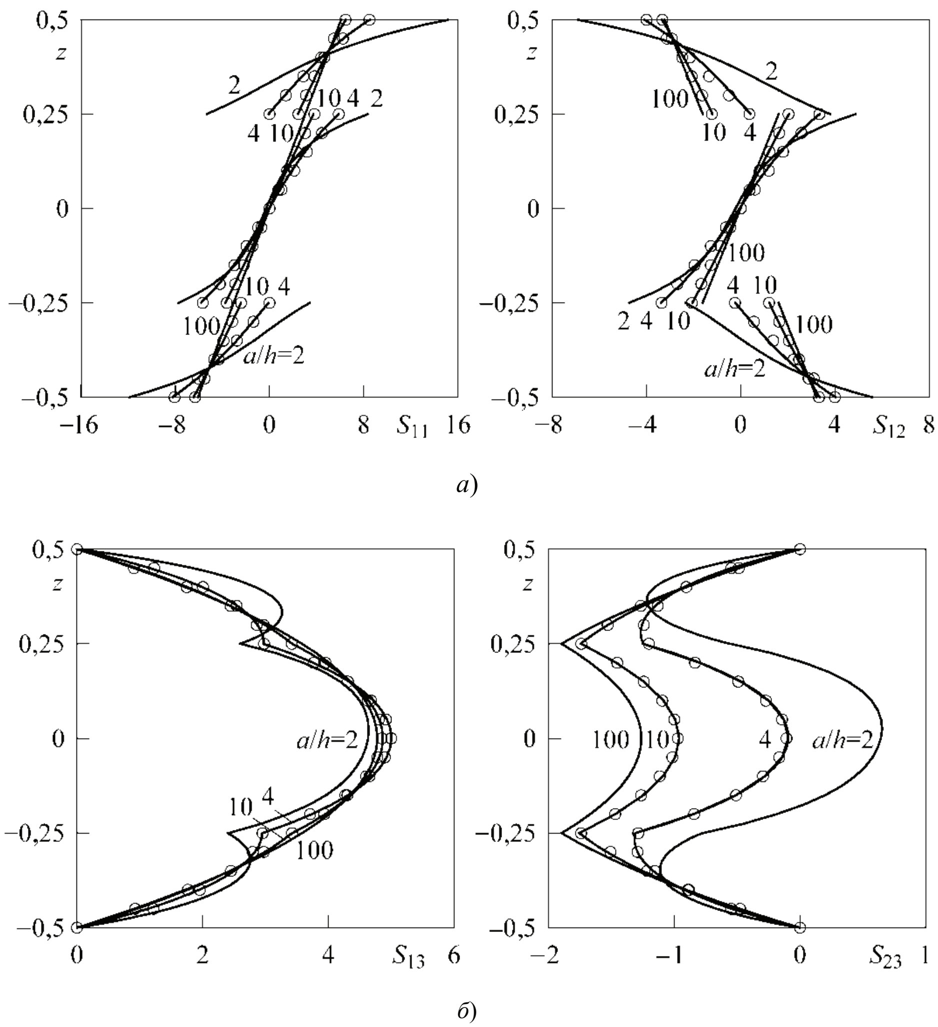

Fig. 2. Distribution of tangential stresses S_{11} and S_{12} (*a*) and transverse stresses S_{13} and S_{23} (*б*) through the thickness of the three-ply plate for $I_1 = I_2 = I_3 = 7$:
—— – present analysis; ○ – Pagano

The data listed in Table show that the SaS technique permits the derivation of exact solutions of plane strain elasticity even for very thick plates with a prescribed accuracy using a large number of SaS. Fig. 2 present the distribution of stresses in the thickness direction for different values of the slenderness ratio *a/h* by choosing seven SaS for each layer. These results demonstrate convincingly the high potential of the proposed layer-wise plate formulation. This is due to the fact that boundary conditions on the bottom and top surfaces and continuity conditions at layer interfaces for transverse shear stresses are satisfied correctly in spite of applying constitutive equations (14). It is also necessary to mention that the proposed SaS method provides the uniform convergence that is impossible with equally spaced SaS.

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Цилиндрический изгиб перекрестно армированных композитных пластин

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Ключевые слова и фразы: метод выборочных поверхностей; слоистая композитная пластина; плоская деформация.

Аннотация: Статья представляет эффективный метод решения плоской задачи теории упругости для слоистых композитных пластин. Метод основан на новой концепции выборочных поверхностей (SaS), разработанной недавно авторами. Согласно этой концепции внутри *n*-го слоя вводятся I_n произвольным образом расположенных SaS параллельных срединной поверхности пластины и в качестве искомых функций выбираются перемещения этих поверхностей. Это дает возможность получать решения плоской задачи теории упругости для толстых и тонких слоистых композитных пластин с заданной точностью, используя достаточно большое число SaS, которые размещаются на поверхностях раздела слоев и в узловых точках полинома Чебышева.

Zylindrische Biegung der kreuzbewehrten Kompositplatten

Zusammenfassung: Der Artikel präsentiert die effective Methode der Lösung der Flachaufgabe der Elastizitätstheorie für die geschichteten Kompositplatten. Die Methode basiert auf der von den Autoren vor kurzem auserarbeiteten neuen Konzeption der stichprobenartigen Oberflächen (SaS). Laut dieser Konzeption werden innen der *n*-Schicht von freiweise angeordneten SaS eingeführt und als Suchfunktionen werden die Umstellungen dieser Oberflächen gewählt. Das gibt die Möglichkeit, die Lösungen der Flachaufgabe der Elastizitätstheorie für die dicken und dünnen geschichteten Kompositplatten mit der angegebenen Genauigkeit zu erhalten, dabei wird es die genug groβe Zahl von SaS, die auf den Oberflächen des Schichtteils und in den Knotenpunkten des Polynomes von Chebyshev, angewandt.

Courbure cylindrique des plattes composites armées de la manière croisée

Résumé: L’article présente une méthode efficace de la solution du problème plat de la théorie de l’élasticité pour les plaques composites fueilletées. La méthode est fondée sur une nouvelle conception des surfaces sélectionnées (SaS), élaborée il n’y a pas longtemps par les auteurs. Suivant cette conception à l’intérieur de la *n*-ème couche sont induits I_n des SaS situées de la manière facultative parallèles à la surface moyenne de la plaque, et en qualité des fonctions recherchées sont choisis les déplacements de ces surfaces. Cela donne la possibilité d’obtenir les solutions du problème plat de la théorie de l’élasticité pour les plaques composites fueilletées fines et grosses avec une précision donnée en utilisant une assez grande quantité de SaS qui sont situées sur les surfaces du domaine des couches et dans les points noueux du polynôme de Tchebichev.

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