Ministry of education and science of the Russian Federation

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MATHEMATICS

FOR

BACHELORS

Limits of functions

The book is intended for students studying economic and engineer specialities. Tbe book is recommended to students of full-time tuition and correspondence tuition

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Reviewer

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Approved by TSTU methodical council

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I. LIMIT OF FUNCTION

1. **Limit of function at $x \to \infty$.**

Let function $y = 2 + \frac{1}{x}$. Let's consider a function at $x \to +\infty$. We will write the table of this function and draw this function graph

x	1	2	10	100	1000
y	2	2,5	2,1	2,01	2,001

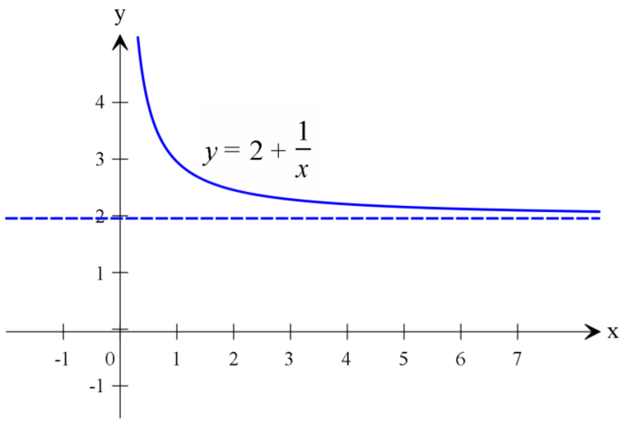


Рис. 1

We remark, that function is being decreased to 2 when the argument *x*, is being increased or we can say that the graph of the function approaches closer to the line $y = 2$. It is possible to describe it other way. For example the function converges to 2. But in mathematics everything should be described only one way.

Let's find the distance from an arbitrary point *M(x,y)* of the function $y = 2 + \frac{1}{x}$ graph to the line $y = 2$: $|f(x) - 2| = \left|2 + \frac{1}{x} - 2\right| = \left|\frac{1}{x}\right| = \frac{1}{|x|}$.

Then the fact that the function $y = 2 + \frac{1}{x}$ at $x \to +\infty$ has limit 2 means that the distance from an arbitrary point *M(x,y)* of the function $y = 2 + \frac{1}{x}$ to the line $y = 2$, can be smaller then any given number for sufficient large *x*.

For example if $x > 10$, then $|f(x) - 2| = \frac{1}{|x|} < \frac{1}{10}$;

if $x > 100$, then $|f(x) - 2| = \frac{1}{|x|} < \frac{1}{100}$.

It is possible to find sufficiently small number $\varepsilon > 0$, that $|f(x) - 2| = \frac{1}{|x|} < \frac{1}{\varepsilon}$, for all $x > \frac{1}{\varepsilon}$.

Let's introduce definition of the function's limit at unrestricted *x*.

Definition. Number A is called the limit of the function $y = f(x)$ at $x \to +\infty$, if for all positive $\varepsilon > 0$ there exists such positive number N that for all x, satisfying $x > N$, we have $\left|f(x) - A\right| < \varepsilon$.

Limit is denoted $\lim\limits_{x \to +\infty} f(x) = A$.

For function $y = 2 + \dfrac{1}{x}$ we have $\lim\limits_{x \to +\infty}\left(2 + \dfrac{1}{x}\right) = 2$.

Geometric sense of limit $x \to +\infty$.

If the function $y = f(x)$ has limit A, then it means, if for every positive number $\varepsilon > 0$ there exists such positive number N, that for all arguments x satisfying $x > N$, we have

$$\left|f(x) - A\right| < \varepsilon. \qquad (1)$$

Let's transform (1), using properties of module:

$$-\varepsilon < f(x) - A < \varepsilon,$$

or

$$A - \varepsilon < f(x) < A + \varepsilon. \qquad (2)$$

Неравенство (2) показывают, что график функции $y = f(x)$ для всех x, превосходящих число N, содержится в полосе, ограниченной прямыми $y = A - \varepsilon$ и $y = A + \varepsilon$.

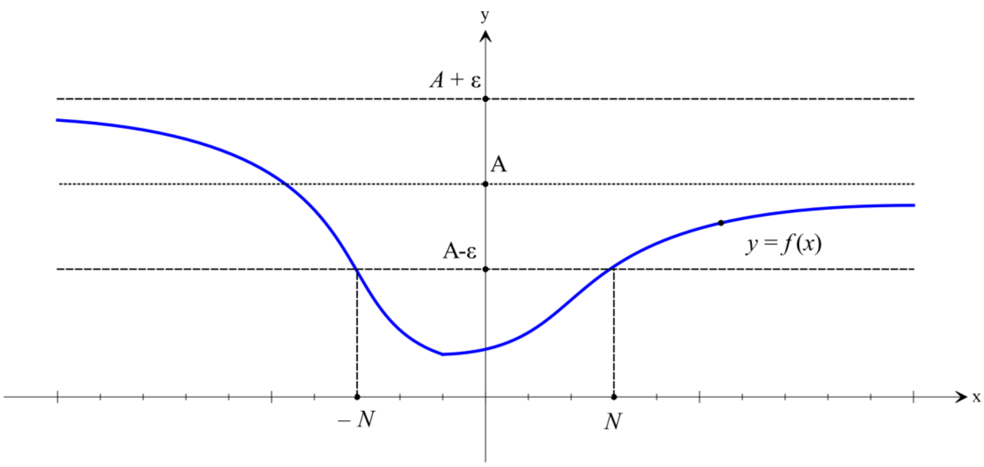


Рис. 2

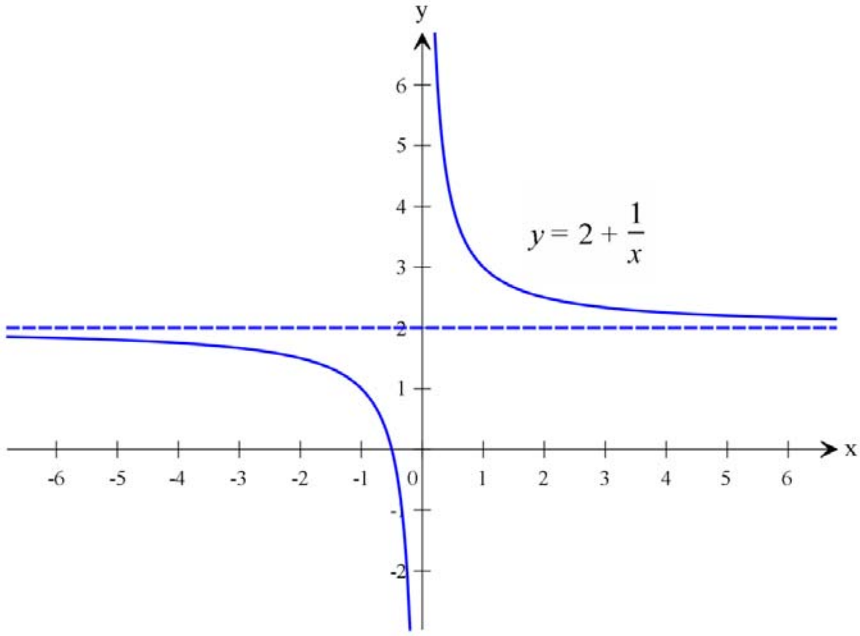
Definition in the case $x \to -\infty$ is introduced analogously.

Definitione. Number A is called a function $y = f(x)$ limit at $x \to -\infty$ if for all positive $\varepsilon > 0$ exists such positive number N that for every values x satisfying the relation $x < -N$, the inequality $\left|f(x) - A\right| < \varepsilon$. is true.

Обозначение: $\lim\limits_{x \to -\infty} f(x) = A$.

Geometric sense of limit at $x \to -\infty$ is analogous to the geometric sense of function's limits at $x \to +\infty$. If $\lim_{x \to -\infty} f(x) = A$, then for every positive number $\varepsilon > 0$ there exists such positive number N that for all x, satisfying the condition $x < -N$, the graph of the function is in the set bounded with the lines $y = A - \varepsilon$ и $y = A + \varepsilon$.

For the function $y = 2 + \frac{1}{x}$ we obtain $\lim_{x \to -\infty}\left(2 + \frac{1}{x}\right) = 2$.



Picture 3

Example. Prove $\lim_{x \to +\infty}\left(\frac{3x-1}{x}\right) = 3$.

Solution. Let's choose an arbitrary number $\varepsilon > 0$ and let's conder $\left|f(x) - A\right|$.

In this case $f(x) = \frac{3x-1}{x}, A = 3$: $\left|f(x) - A\right| = \left|\frac{3x-1}{x} - 3\right| = \left|\frac{-1}{x}\right| = \frac{1}{|x|}$.

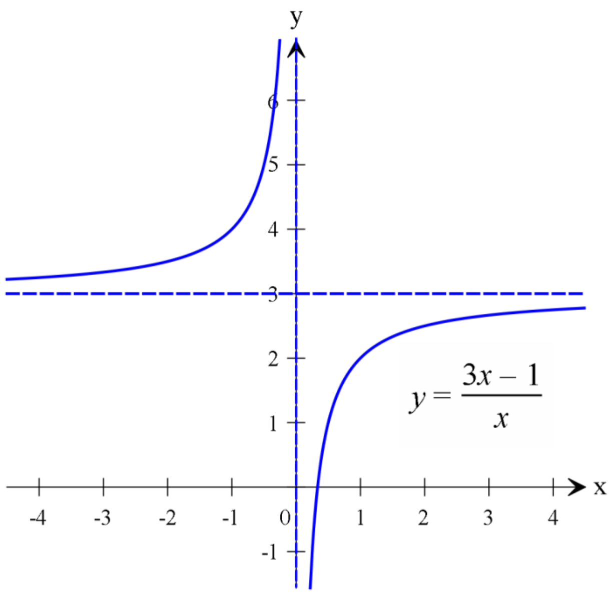


Рис. 4

It is sufficient $x > \frac{1}{\varepsilon}$ to satisfy the inequality $\left|f(x) - A\right| = \left|\frac{3x-1}{x} - 3\right| = \frac{1}{|x|} < \varepsilon$ The number x is positive because we consider $x \to +\infty$ We may suppose $N = \frac{1}{\varepsilon}$. So we obtain for all $\varepsilon > 0$ there exists positive such number $N = \frac{1}{\varepsilon}$ that for all x, satisfying the condition $x > N = \frac{1}{\varepsilon}$, the inequality $\left|\frac{3x-1}{x} - 3\right| < \varepsilon$ is true. It means that $\lim\limits_{x\to+\infty}\left(\frac{3x-1}{x}\right) = 3$.

See graph of the function on the pic 4.

2. Limit of function at $x \to a$.

Let's introduce the definition of limit at a point. Let's consider the function $y = f(x)$ defined on an interval, containing point $x = a$.

Definition. The number A is called a limit of the function $y = f(x)$ at $x \to a$ (or at the point a), if for an arbitrary number $\varepsilon > 0$ there exists such number $\delta > 0$ that for all x satisfying the condition

$$0 < |x - a| < \delta, \quad (3)$$

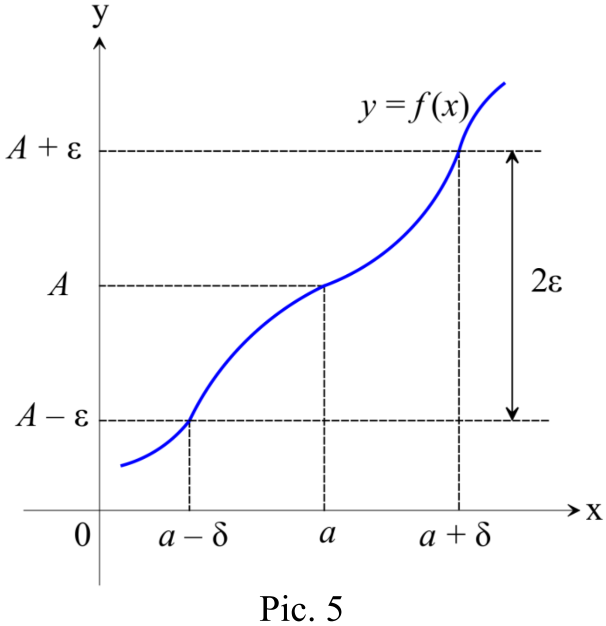
the inequality

$$|f(x) - A| < \varepsilon. \quad (4)$$

is true.

The limit of the function $y = f(x)$ at the point a is denoted $\lim\limits_{x\to a} f(x) = A$.

The geometric sense of this definition is following. The inequalities (3) mean that the distance between the points x and a is not larger then δ, hence it is contained in the interval $(a-\delta; a+\delta)$. The inequality (4) means that the values of the function $y = f(x)$ are in the interval $(A-\varepsilon; A+\varepsilon)$. The dots of the graph $y = f(x)$ must be in the set bounded with the lines $y = A - \varepsilon$ и $y = A + \varepsilon$. See pic. 5.



Pic. 5

Example. Find limit of the function $y=3x-1$ at $x\to 1$

Usng the graph of the (pic. 6) we can see thet if если $x \to 1$ from any side then points $M(x, y)$ of the graph converge to the point $M(1, 2)$ hence we may suppose $\lim_{x\to 1}(3x-1)=2$. Let's prove it. Let's take an arbitrary number $\varepsilon > 0$ and consider the conditions when the inequality $|(3x–1) – 2|<\varepsilon$ or $|3x–3| < \varepsilon$ is true. Hence

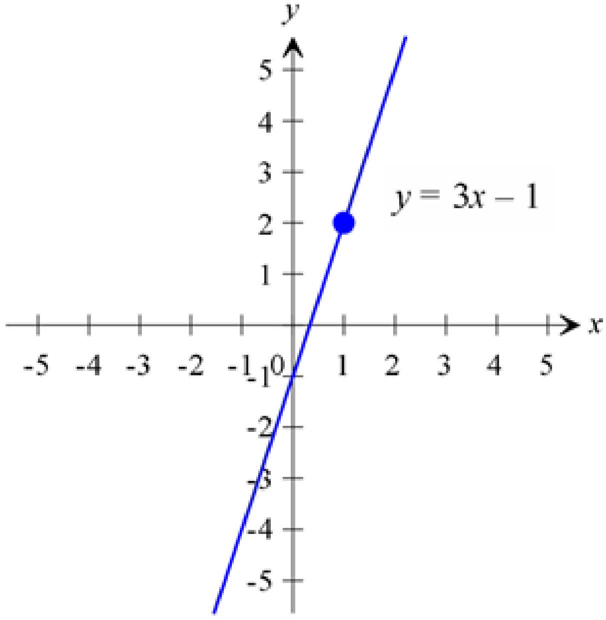


Рис. 6

So of we suppose $\delta = \frac{\varepsilon}{3}$, then for all x satisfying the inequality $|x– 1|<\delta$ the inequality $|(3x-–1) –2| < \varepsilon$ is true. According to the definition of limit it means that 2 is the limit of the function $y=3x-1$ at $x\to 1$.

In the considered example the limit of the function at $x \to 1$ equals to the value of the function at $x=1$. It is true not for all functions but only for those which have smooth line as a graph. When we draw the graph of a function $y = f(x)$ we see that when the argument x converges to the point a then the value $y = f(x)$ converges to $f(a)$.

Let's give the strict definition of continuous function. Let us have the function $y = f(x)$.

Definition. The function $y = f(x)$ is called continuous at the point x_0 if it is defined at this point and at the vicinity of this point $\lim_{x\to x_0} f(x) = f(x_0)$.

Remark. When the limit of a continuous function at the point x_0 is calculated it is ought to substitute the value x_0 to the function then we obtain the limit at the point.

Example. Calculate $\lim_{x\to -2} x^3$.

The function is continuous at the point $x=-2$. Then

$$\lim_{x\to -2} x^3 = \lim_{x\to -2}(-2)^3 = \lim_{x\to -2}(-8) = -8.$$

3. **Unilateral limits.**

Let us have the function $y = \frac{1}{x}$.

And let's consider behaviour of the function near the point $x = 0$ (pic. 7.)

When the argument x к нулю from right-hand side then values of the function are being increased., $y \to +\infty$, but

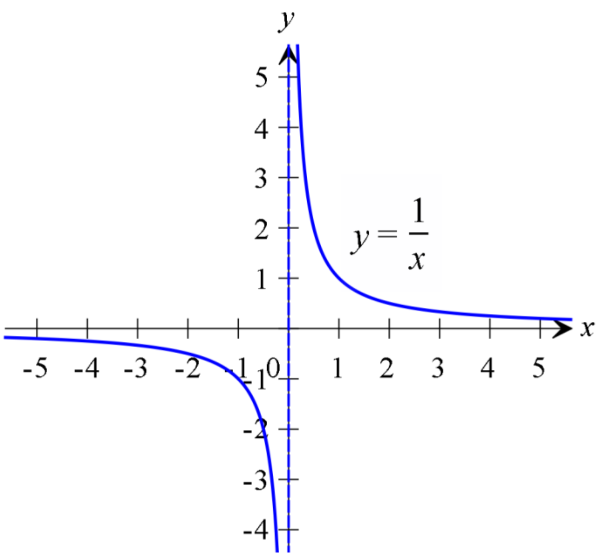


Рис. 7

when x converges to zero from left-hand side the function is being decreased $y \to -\infty$. In this case we see that limit depends on method of converging.

Therefore unilateral limits are considered.

Definition. Number A_1 is called as a function $y = f(x)$ limit on the left at the point a, if for every number $\varepsilon > 0$ there is such number $\delta > 0$ that at $x \in (a-\delta; a)$ the inequality $\left|f(x) - A_1\right| < \varepsilon$ is true.

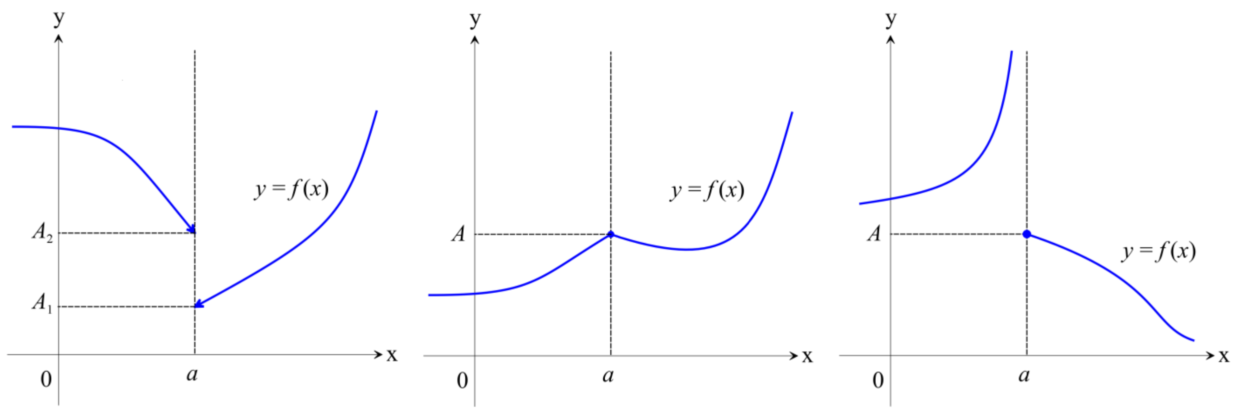
Notation $\lim\limits_{x \to a-0} f(x) = A_1$.

Definition. Number A_2 is called a limit of the function $y = f(x)$ on the right at the point a, if for every number $\varepsilon > 0$ exists such $\delta > 0$ that for $x \in (a; a+\delta)$ the inequality $\left|f(x) - A_2\right| < \varepsilon$ is true.

Notation: $\lim\limits_{x \to a+0} f(x) = A_2$.

Function limits on the right and on the left are called unilateral limits. It is possible to prove that if unilateral limits equal each other $A_1 = A_2 = A$ (pic. 8, b)) then limit of a function at the point a exists and equals unilateral limits $\lim\limits_{x \to a} f(x) = A$.

It unilateral limits are different $A_1 \neq A_2$ or though one of them does not exist then limit of a function at the point a does not exist too (pic..8, a), c)).



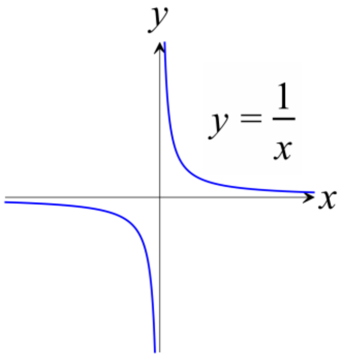
a) b) c)

Pic. 8

From function limit we obtain that limit of a constant equals this constant $\lim\limits_{x \to a} c = c$, where c-const.

Let's consider the graphs of some elementary functions that will be useful for investigating functions.

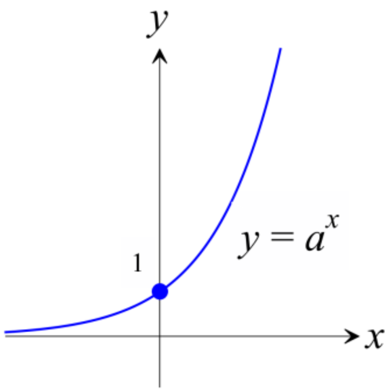
Hyperbola

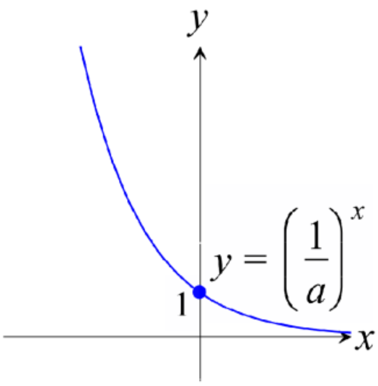


$$\lim_{x\to 0+0}\frac{1}{x}=+\infty,\quad \lim_{x\to 0-0}\frac{1}{x}=-\infty,$$

$$\lim_{x\to +\infty}\frac{1}{x}=0,\quad \lim_{x\to -\infty}\frac{1}{x}=0.$$

Exponent $y=a^x$





$a>1$:

$\lim_{x\to -\infty} a^x = 0$,

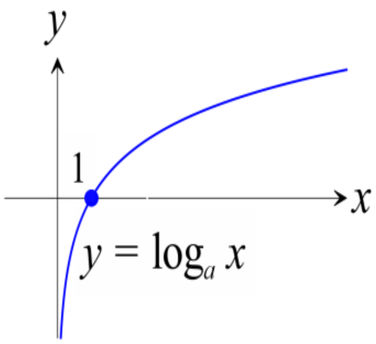
$\lim_{x\to +\infty} a^x = +\infty$;

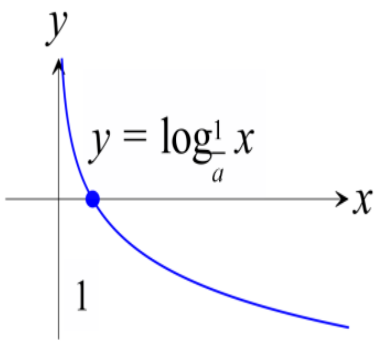
$0<a<1$:

$\lim_{x\to -\infty} a^x = +\infty$,

$\lim_{x\to +\infty} a^x = 0$.

Logarithm $y=\log_a x$





$a>1$:

$\lim_{x\to +\infty}\log_a x = +\infty$,

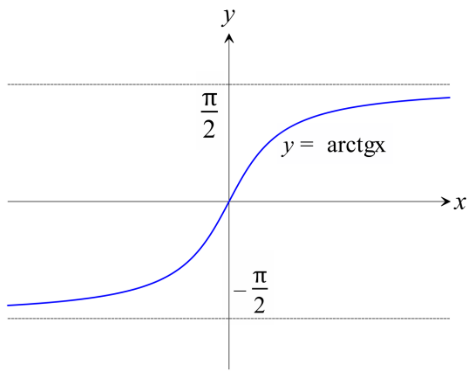
$\lim_{x\to 0+0}\log_a x = -\infty$;

$0<a<1$:

$\lim_{x\to +\infty}\log_a x = -\infty$,

$\lim_{x\to 0+0}\log_a x = +\infty$.

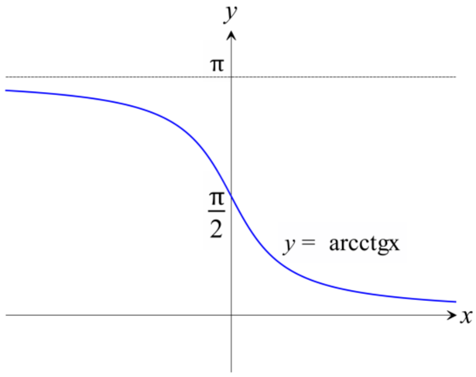
Inverse trigonometric functions



$y = arctgx$:

$\lim\limits_{x\to+\infty} arctgx = \frac{\pi}{2}$,

$\lim\limits_{x\to-\infty} arctgx = -\frac{\pi}{2}$;



$y = arcctgx$:

$\lim\limits_{x\to+\infty} arcctgx = 0$,

$\lim\limits_{x\to-\infty} arcctgx = \pi$.

4. Infinitesimal functions and there properties.

Definition. Function $\alpha = \alpha(x)$ is called infinitesimal at $x \to a$ (or $x \to \infty$), if $\lim\limits_{x\to a} \alpha(x) = 0$ (or $\lim\limits_{x\to\infty} \alpha(x) = 0$).

Fuction $\alpha(x) = (x+1)^2$ is infinitesimal at $x \to -1$ because $\lim\limits_{x\to-1} \alpha(x) = \lim\limits_{x\to-1} (x+1)^2 = 0$. Function $\alpha(x) = \frac{1}{x}$ is also infinesimal, at $x \to \infty$ because .

Properties of infinitesimal functions.

1. Algebraic sum of finite number of infinitesimal functions is also an infinitesimal.

Corollary 1. Product of two infinitesimals is an infinitesimal function.

Corollary 2. Product of an infinitesimal function and a constant is an infinitesimal.

Quotient of division an infinitesimal by function having nonzero limit is an infinitesimal.

Comparison of infinitesimal functions.

The quotient of two infinitesimal functions is determined unambiguously. Different cases are possible.

Let $\alpha(x)$ and $\beta(x)$ be infinitesimal functions at $x \to a$. Then if $\lim\limits_{x\to a}\frac{\alpha(x)}{\beta(x)} = 0$, then $\alpha(x)$ is called an infinitesimal of higher order of smallness with respect to $\beta(x)$: $\alpha(x) = o(\beta(x))$;

1) $\lim\limits_{x\to a}\frac{\alpha(x)}{\beta(x)} = A \neq 0$, then $\alpha(x)$ and $\beta(x)$ infinitesimal of the same order; if $A = 1$ then they are called equivalent infinitesimal: $\alpha(x) \sim \beta(x)$;
2) $\lim\limits_{x\to a}\frac{\alpha(x)}{\beta(x)} = \infty$, then $\beta(x)$ is called an infinitesimal of higher order of smallness with respect to c $\alpha(x)$: $\beta(x) = o(\alpha(x))$.

So a relation of two infinitesimal is an indefinite value and is denoted $\left[\frac{0}{0}\right]$.

To solve the indeterminancy $\left[\frac{0}{0}\right]$ it is often useful to replace an infinitesimal with equivalent one. if $\alpha(x) \sim \alpha_1(x)$, $\beta(x) \sim \beta_1(x)$ and $x \to a$ and there exists $\lim\limits_{x\to a}\frac{\alpha_1(x)}{\beta_1(x)}$, then $\lim\limits_{x\to a}\frac{\alpha(x)}{\beta(x)}$ also exists and

$$\lim_{x\to a}\frac{\alpha(x)}{\beta(x)} = \lim_{x\to a}\frac{\alpha_1(x)}{\beta_1(x)}.$$

Some important equivalences are given below. For example, at $x \to 0$:

$\sin x \sim x$;	$e^x - 1 \sim x$;
$tgx \sim x$;	$a^x - 1 \sim x \cdot \ln a$;
$\arcsin x \sim x$;	$\ln(1+x) \sim x$;
$arctgx \sim x$;	$\log_a(1+x) \sim x \cdot \log_a e$;
$1 - cosx \sim \frac{x^2}{2}$;	$(1+x)^k - 1 \sim k \cdot x, k > 0,$
	$\sqrt{1+x} - 1 \sim \frac{x}{2}$.

We will consider using of this equivalences in the next section.

5. Infinitely large functions and their properties.

Definition. Function $y = f(x)$ is called infinitely large at $x \to a$, if for every positive number *N*, exists such number $\delta > 0$ that for all *x*, satisfying $0 < |x - a| < \delta$ the inequality $|f(x)| > N$ .is true.

The example of infinitely large function is $y = \frac{1}{x}$ at $x \to 0$ (рис.7).

Infinitely large function hasn't finite limit at $x \to a$, its limit is infinity. If the function has only positive values then it is ought to write $\lim\limits_{x\to a} f(x) = +\infty$, if only negative then $\lim\limits_{x\to a} f(x) = -\infty$.

Properties of infinitely large functions

1. The sum of infinitely large function and limited function is an infinitely large function.

Remark. Sum of infinitely large functions is an infinitely large function. $[\infty + \infty] \to +\infty$; the result of infinitely large functions subtraction is an indefinite value $[\infty - \infty] \to ?$ solving such indefinites will be considered below.

The product of infinitely large function and a function with nonzero limit is an infinitely large function.

2. The ratio of infinitely large function and a function having limit at a point is an infinitely large function. R

Remark. The result of division infinitely large function with infinitely large functions an indefinite value and id denoted $\left[\frac{\infty}{\infty}\right]$ abd the method of solving such indefinites will be considered below.

Relations between the infinitesimals and infinitely large functions.

If the function $\alpha = \alpha(x)$ converges to zero at $x \to a$ (or $x \to \infty$) and does not transform into zero then the function $f(x) = \frac{1}{\alpha(x)}$ converges to infinity.

6. Basic theorems about function limits

Esides function $y = f(x)$ can not have more than one limit at $x \to a$.

1) If every function of $f(x)$ and $\varphi(x)$ has limit at $x \to a$, then sum, subtraction and product of this functions also have limits and

$$\lim_{x\to a}\left[f(x) \pm \varphi(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} \varphi(x);$$

$$\lim_{x\to a}\left[f(x) \cdot \varphi(x)\right] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} \varphi(x).$$

Besides if $\lim\limits_{x\to a} \varphi(x) \neq 0$ then the ratio $\frac{f(x)}{\varphi(x)}$ has limit and

$$\lim_{x\to a} \frac{f(x)}{\varphi(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} \varphi(x)}.$$

Corollary 1. Constant multiplier can be taken out of limit

$$\lim_{x\to a}\left[cf(x)\right] = c \cdot \lim_{x\to a} f(x).$$

Corollary 2. If $\lim_{x\to a} f(x) = A$ and n – is a natural number then

$$\lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n,$$

in particular

$$\lim_{x\to a}(x^n) = (\lim_{x\to a} x)^n = a^n .$$

2) If $\lim_{x\to x_0}\varphi(x) = u_0$, but $\lim_{u\to u_0} f(u) = A$, то $\lim_{x\to x_0} f(\varphi(x)) = A$.

7. Remarkable limits.

Let's consider the function $f(x) = \dfrac{\sin x}{x}$, the is not defined at $x = 0$. There exists the theorem proving that limit of the function exists

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 . \qquad (*)$$

This limit is called: *first remarkable limit*.

It follows from the formula (*) that $\lim_{x\to 0}\dfrac{tgx}{x} = 1$.

This formulae are used to solve the indeterminancy $\left[\dfrac{0}{0}\right]$ in the case if the function contains trigonometric function.

The second remarkable limit is used to solve indefinites $\left[1^\infty\right]$.

Let's consider the function $f(x) = (1+\dfrac{1}{x})^x$. Here is the indeterminancy $\left[1^\infty\right]$.

There it the theorem which states that limit of the function $f(x) = (1+\dfrac{1}{x})^x$ at $x\to\infty$ exists and equals *e*:

$$\lim_{x\to\infty}(1+\frac{1}{x})^x = e .$$

The second remarkable limit has else one form

$$\lim_{y\to 0}(1+y)^{\frac{1}{y}} = e .$$

The examples are given in the next section

II. Examples of typical problems solutions.

1. Let's consider the tasks $\lim\limits_{x\to\infty}\frac{f(x)}{\varphi(x)}$ with indeterminancy of the formc $\left[\frac{\infty}{\infty}\right]$, where $f(x)$ and $\varphi(x)$ in the general case are complicated degree or exponential function

а) $\lim\limits_{x\to\infty}\frac{3x^2-4x+1}{7-4x^2}=\left[\frac{\infty}{\infty}\right].$

Let's find the greatest degree of x in the numerator в and in the denominator. It is x^2. Let's divide the numerator and the denominator by x^2:

$$\lim_{x\to\infty}\frac{\frac{3x^2}{x^2}-\frac{4x}{x^2}+\frac{1}{x^2}}{\frac{7}{x^2}-\frac{4x^2}{x^2}}=\lim_{x\to\infty}\frac{3-\frac{4}{x}+\frac{1}{x^2}}{\frac{7}{x^2}-4}=-\frac{3}{4}.$$

б) $$\lim_{x\to\infty}\frac{7x^5+4x^3}{3x^2-2x+1}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{\frac{7x^5}{x^5}+\frac{4x^3}{x^5}}{\frac{3x^2}{x^5}-\frac{2x}{x^5}+\frac{1}{x^5}}=$$

$$=\lim_{x\to\infty}\frac{7+\frac{4}{x^2}}{\frac{3}{x^2}-\frac{2}{x^4}+\frac{1}{x^5}}=\frac{7}{0}=\infty.$$

в) $$\lim_{x\to\infty}\frac{4x-2}{x^4+x^3-2x+1}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{\frac{4x}{x^4}-\frac{2}{x^4}}{\frac{x^4}{x^4}+\frac{x^3}{x^4}-\frac{2x}{x^4}+\frac{1}{x^4}}=$$

$$=\lim_{x\to\infty}\frac{\frac{4}{x^3}-\frac{2}{x^4}}{1+\frac{1}{x}-\frac{2}{x^3}+\frac{1}{x^4}}=\frac{0}{1}=0.$$

г) $$\lim_{x\to\infty}\frac{\sqrt[3]{27x^3+4x^2-2x+1}+5x-3}{2-7x}=\left[\frac{\infty}{\infty}\right].$$

Let's find greatest degree of x in the numerator and in the denominator. This is x^1.

Let's consider $\sqrt[3]{27x^3+4x^2-2x+1}$ (let's bring number x^3 under root) $\sqrt[3]{x^3\left(27+\frac{4}{x}-\frac{2}{x^2}+\frac{1}{x^3}\right)}=x\sqrt[3]{27+\frac{4}{x}-\frac{2}{x^2}+\frac{1}{x^3}}$.

Let's substitute the expression to this limit and divide the numerator and the denominator by x^1 :

$$\lim_{x\to\infty}\frac{\frac{x\sqrt[3]{27+\frac{4}{x}-\frac{2}{x^2}+\frac{1}{x^3}}}{x}+\frac{5x}{x}-\frac{3}{x}}{\frac{2}{x}-\frac{7x}{x}}=$$

$$=\lim_{x\to\infty}\frac{\sqrt[3]{27+\frac{4}{x}-\frac{2}{x^2}+\frac{1}{x^3}}+5-\frac{3}{x}}{\frac{2}{x}-7}=-\frac{8}{7}=-1\frac{1}{7}.$$

2. Let's consider the tasks $\lim\limits_{x\to x_0}\frac{f(x)}{\varphi(x)}$ with indeterminancy of the form $\left[\frac{0}{0}\right]$. In this case it is necessary to decompose into multiples numerator and denominator to multiply the numerator and the denominator pf the fraction the same expression.

a) $\lim\limits_{x\to -3}\frac{x^2-9}{x^2+7x+12}=\lim\limits_{x\to -3}\frac{(x-3)(x+3)}{(x+3)(x-4)}=\frac{-3-3}{-3+4}=-6$.

[We factorize the numerator according to the formula $x^2-y^2=(x-y)(x+y)$].

We factorize the expression
$$x^2+7x+12=(x+3)(x+4)$$
$$D=49-48=1$$
$$x_1=\frac{-7+1}{2}=-3;\ x_2=\frac{-7-1}{2}=-4$$

б) $\lim\limits_{x\to 4}\frac{\sqrt{25-x^2}-3}{x-4}=\left[\frac{0}{0}\right]=\lim\limits_{x\to 4}\frac{\left(\sqrt{25-x^2}-3\right)\left(\sqrt{25-x^2}+3\right)}{\left(\sqrt{25-x^2}+3\right)(x-4)}=$

(Let's multiply the numerator and the denominator with the expression interfaced the numerator: $\sqrt{25-x^2}+3$. In the numerator we will obtain $\left(\sqrt{25-x^2}-3\right)\left(\sqrt{25-x^2}+3\right)$).

$$=\lim_{x\to 4}\frac{25-x^2-9}{\left(\sqrt{25-x^2}+3\right)(x-4)}=\lim_{x\to 4}\frac{16-x^2}{\left(\sqrt{25-x^2}+3\right)(x-4)}=$$

$$=\lim_{x\to 4}\frac{(4-x)(4+x)}{\left(\sqrt{25-x^2}+3\right)(x-4)}=-\frac{8}{9}.$$

в) $\lim_{x\to 2}\frac{\sqrt{3x-2}-2}{\sqrt{x}-\sqrt{2}}=\left[\frac{0}{0}\right]=\lim_{x\to 2}\frac{\left(\sqrt{3x-2}-2\right)\left(\sqrt{3x-2}+2\right)\left(\sqrt{x}+\sqrt{2}\right)}{\left(\sqrt{x}-\sqrt{2}\right)\left(\sqrt{3x-2}+2\right)\left(\sqrt{x}+\sqrt{2}\right)}=$

It is necessary to multiply the numerator and the denominator the interfaced expression: $\left(\sqrt{3x-2}+2\right)\left(\sqrt{x}+\sqrt{2}\right)$

$$=\lim_{x\to 2}\frac{(3x-2-4)\left(\sqrt{x}+\sqrt{2}\right)}{(x-2)\left(\sqrt{3x-2}+2\right)}=\lim_{x\to 2}\frac{3(x-2)\left(\sqrt{x}+\sqrt{2}\right)}{(x-2)\left(\sqrt{3x-2}+2\right)}=\frac{3\left(\sqrt{2}+\sqrt{2}\right)}{4}=$$

$$=\frac{6\sqrt{2}}{4}=\frac{3}{2}\sqrt{2}.$$

3. Let's consider the indeterminancy $[\infty-\infty]$.

a) $\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{2x+8}{x^3-8}\right)=[\infty-\infty]$.

We have the indeterminancy $[\infty-\infty]$.

Let's reduce the fractions to common denominator

$$\lim_{x\to 2}\frac{x^2+2x+4-2x-8}{(x-2)\left(x^2+2x+4\right)}=\lim_{x\to 2}\frac{x^2-4}{(x-2)\left(x^2+2x+4\right)}=$$

$$=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)\left(x^2+2x+4\right)}=\frac{4}{12}=\frac{1}{3}.$$

б) $\lim_{x\to\infty}\left(\sqrt{x^2-2x-5}-x\right)=[\infty-\infty]$.

We have indeterminancy of the form $[\infty-\infty]$.

Let's multiply and divide the function the expression interfaced it.

$$\lim_{x\to\infty}\frac{\left(\sqrt{x^2-2x-5}-x\right)\left(\sqrt{x^2-2x-5}+x\right)}{\sqrt{x^2-2x-5}+x}=\lim_{x\to\infty}\frac{x^2-2x-5-x^2}{\sqrt{x^2-2x-5}+x}=$$

$$=\lim_{x\to\infty}\frac{-2x-5}{\sqrt{x^2-2x-5}+x}.$$

Let's find the greatest degree of the fraction This is x^1, hence:

$$\lim_{x\to\infty}\frac{-2-\frac{5}{x}}{\sqrt{1-\frac{2}{x}-\frac{5}{x^2}}+1}=-1.$$

Also at $x\to\infty$ $\frac{5}{x};\frac{1}{x};\frac{5}{x^2}$ – are infin.itesimals

Remarkable limits $\lim\limits_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x=e$.

4. Let's consider the tasks with the indeterminancy $\left[1^\infty\right]$.

a) $\lim\limits_{x\to\infty}\left(\frac{2x^2-3}{2x^2+1}\right)^{-3x^2}$.

We have indeterminancy $\left[1^\infty\right]$,

$$\lim_{x\to\infty}\frac{2x^2-3}{2x^2+1}=\lim_{x\to\infty}\frac{2-\frac{3}{x^2}}{2+\frac{1}{x^2}}=\frac{2}{2}=1.$$

Find the integer part of the fraction:

$$\frac{2x^2-3}{2x^2+1}=\frac{\left(2x^2+1\right)-1-3}{2x^2+1}=\frac{\left(2x^2+1\right)-4}{2x^2+1}=1-\frac{4}{2x^2+1}=1+\left(\frac{-4}{2x^2+1}\right)$$

$\frac{-4}{2x^2+1}=\alpha(x)$ – is an infinitesimal at $x\to\infty$.

We will multiply the degree $\left(\alpha(x)\frac{1}{\alpha(x)}\right)$, it retains the equality

$$\lim_{x\to\infty}\left(1+\left(\frac{-4}{2x^2+1}\right)^{\frac{2x^2+1}{-4}\cdot\left(\frac{-4}{2x^2+1}\right)}\right)^{-3x^2} =$$

$$= \lim_{x\to\infty}\left(1+\left(\frac{-4}{2x^2+1}\right)^{\frac{2x^2+1}{-4}}\right)^{\frac{12x^2}{2x^2+1}} = \lim_{x\to\infty} e^{\frac{12x^2}{2x^2+1}} = e^{\lim\limits_{x\to\infty}\frac{12x^2}{2x^2+1}} = e^6.$$

$$\left[\lim_{x\to\infty}\frac{12x^2}{2x^2+1} = \lim_{x\to\infty}\frac{12}{2+\frac{1}{x^2}} = 6\right].$$

б) $\lim\limits_{x\to 0}\left(\frac{x-1}{2x-1}\right)^{\frac{3}{x}}$.

We have the indeterminancy $\left[1^\infty\right]$, because $\lim\limits_{x\to 0}\frac{x-1}{2x-1} = 1$.

Let's find the integer part $\frac{x-1}{2x-1} = \frac{(2x-1)-2x+1+x-1}{2x-1} = \frac{(2x-1)-x}{2x-1} =$

$= 1+\left(\frac{-x}{2x-1}\right)$.

$$\lim_{x\to 0}\left(1+\left(\frac{-x}{2x-1}\right)\right)^{\frac{2x-1}{-x}\cdot\left(\frac{-x}{2x-1}\right)\cdot\frac{3}{x}} = \lim_{x\to 0}\left(\left(1+\left(\frac{-x}{2x-1}\right)\right)^{\frac{2x-1}{-x}}\right)^{\frac{-3}{2x-1}} =$$

$= e^{\lim\limits_{x\to 0}\frac{-3}{2x-1}} = e^3$. ($\lim\limits_{x\to 0}\left(1+\left(\frac{-x}{2x-1}\right)\right)^{\frac{2x-1}{-x}} = e$).

в) $\lim\limits_{x\to 0}\frac{\ln(x+3)-\ln 3}{5x}$.

We have the indeterminancy $\left[\frac{0}{0}\right]$.

Let's use the logarithm property ($\log_a b - \log_a c = \log_a \frac{b}{c}$;

$n\log_a x = \log_a x^n$).

$$\lim_{x\to 0}\frac{\ln\frac{x+3}{3}}{5x} = \lim_{x\to 0}\left(\frac{1}{5x}\cdot\ln\frac{x+3}{3}\right) = \lim_{x\to 0}\ln\left(\frac{x+3}{3}\right)^{\frac{1}{5x}} = \lim_{x\to 0}\ln\left(1+\frac{x}{3}\right)^{\frac{1}{5x}}.$$

Logarithm is a continuous function, hence the symbols lim and ln can be replaced

$$\ln\lim_{x\to 0}\left(1+\frac{x}{3}\right)^{\frac{1}{5x}} = \ln\lim_{x\to 0}\left(1+\frac{x}{3}\right)^{\frac{3}{x}\cdot\frac{x}{3}\cdot\frac{1}{5x}} = \ln e^{1}15 = \frac{1}{15}.$$

5. $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1.$

а) Calculate $\lim\limits_{x\to 0}\frac{\mathrm{tg}x}{x}$.

$$\lim_{x\to 0}\frac{\mathrm{tg}x}{x} = \left[\frac{0}{0}\right] = \lim_{x\to 0}\frac{\sin x}{x\cos x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x} = 1.$$

Так как $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$; $\lim\limits_{x\to 0}\frac{1}{\cos 0} = 1.$

б) $\lim\limits_{x\to 0}\frac{\arcsin x}{x}$.

We have the indeterminancy $\left[\frac{0}{0}\right]$.

We will substitute $\arcsin x = y \Rightarrow x = \sin y$ at $x \to 0$; $y \to 0$, then we obtain

$$\lim_{x\to 0}\frac{\arcsin x}{x} = \lim_{y\to 0}\frac{y}{\sin y} = 1.$$

в) $\lim\limits_{x\to 0}\frac{\sin 2x + \sin 8x}{4x} = \left[\frac{0}{0}\right] = \lim\limits_{x\to 0}\frac{2\sin 5x\cdot\cos 3x}{4x} =$

(We applied the formula $\sin x+\sin y=2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$ for the numerator.
).

$$=\lim_{x\to 0}\frac{2\frac{\sin 5x}{5x}\cdot 5x\cdot\cos 3x}{4x}=\frac{10}{4}=\frac{5}{2}.$$

(because cos (3·0) = cos 0 = 1).

Tasks for training

1. $\lim_{x\to\infty}\frac{4x^3-8x^5}{4x^7+2}$; $\lim_{x\to\infty}\frac{7x^8-2x+1}{3-2x}$; $\lim_{x\to\infty}\frac{\sqrt[3]{9x^3+2x}-6x+2}{3-x}$

$\lim_{x\to\infty}\frac{\sqrt[3]{64x^3-27x^2+1}-x+1}{2x+4}$; $\lim_{x\to\infty}\left(\sqrt{2x^2-5x+1}-\sqrt{2x^2+4x}\right)$;

$\lim_{x\to\infty}\frac{x^3+2x-3}{4-7x^3}$; $\lim_{x\to\infty}\left(\sqrt{4x^2-2}-\sqrt{4x^2+5x-6}\right)$.

2. $\lim_{x\to 2}\frac{x^2-2x}{x^2-4x+4}$; $\lim_{x\to 3}\frac{x^3-27}{x^2-9}$; $\lim_{x\to -3}\frac{x^2+7x+12}{x^2+6x+9}$;

$\lim_{x\to 2}\frac{x^2+x-6}{2x^2-x-6}$; $\lim_{x\to 3}\frac{x^2-5x+6}{(x-3)^2}$; $\lim_{x\to 1}\frac{x^2-3x+2}{x^2+x-2}$; $\lim_{x\to 4}\frac{x^2-5x+4}{x^2-16}$.

3. $\lim_{x\to 2}\frac{\sqrt{x-1}-1}{x-2}$; $\lim_{x\to 5}\frac{\sqrt{3x^2-11}-8}{x-5}$; $\lim_{x\to 7}\frac{x^2-49}{\sqrt{x}-\sqrt{7}}$;

$\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{\sqrt{x+3}-\sqrt{3}}$; $\lim_{x\to 3}\frac{\sqrt{x+1}-2}{5-\sqrt{22+x}}$; $\lim_{x\to -3}\frac{\sqrt{2x^2+7}-5}{x+3}$;

$\lim_{x\to 0}\frac{\sqrt{3x+7}-\sqrt{7}}{\sqrt{5x+3}-\sqrt{3}}$.

4. $\lim_{x\to\infty}\left(\frac{2x+1}{2x-5}\right)^{-x+4}$; $\lim_{x\to\infty}\left(\frac{x-2}{x+8}\right)^{4-2x}$; $\lim_{x\to\infty}\left(\frac{x^2+1}{x^2-2x+4}\right)^{\sqrt{x}}$;

$\lim\limits_{x\to\infty}\left(\frac{4x+3}{7x-4}\right)^{\frac{1}{x}}$; $\lim\limits_{x\to\infty}\left(\frac{x^2+4x-1}{x^2-2x+8}\right)^{-3x+4}$; $\lim\limits_{x\to\infty}\left(\frac{7x+1}{3x-2}\right)^{2-\sqrt{x}}$.

5. $\lim\limits_{x\to 0}\frac{\mathrm{tg}10x}{\sin 3x}$; $\lim\limits_{x\to 0}\frac{\mathrm{tg}3x}{\cos 2x}$; $\lim\limits_{x\to 0}\frac{\cos x-\cos 2x}{x^2}$;

$\lim\limits_{x\to 0}\frac{\sin x+\sin 4x}{x^2}$ $\lim\limits_{x\to 0}\frac{1-\cos 4x}{2x^2}$.

III. Tasks for individual training

Find the limits

I

1. a) $\lim\limits_{x\to -1}\left(\frac{1}{x+1}-\frac{1}{x^2-1}\right)$; б) $\lim\limits_{x\to -8}\frac{2x^2+15x-8}{3x^2+25x+8}$.

2. a) $\lim\limits_{x\to 0}\left(\frac{1}{2x^2-x}-\frac{1}{x^2-x}\right)$; б) $\lim\limits_{x\to -3}\frac{4x^2+3x+15}{x^2-6x-27}$.

3. a) $\lim\limits_{x\to 1}\left(\frac{1}{x^2-x}-\frac{3}{x^3-1}\right)$; б) $\lim\limits_{x\to 3}\frac{x^2-9}{6x^2-16x-6}$.

4. a) $\lim\limits_{x\to\infty}\frac{5x^5-x+2}{2x^5+3x^2+1}$; б) $\lim\limits_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{\sqrt{x+3}-\sqrt{3}}$.

5. a) $\lim\limits_{x\to 0}\frac{\sqrt{4-x}-\sqrt{4+x}}{3x}$; б) $\lim\limits_{x\to 2}\frac{x^2+x-6}{2x^2-x-6}$.

6. a) $\lim\limits_{x\to\infty}\frac{3x^3+2x^2-x+1}{x^3+3x^2+4x-6}$; б) $\lim\limits_{x\to 0}\frac{3-\sqrt{x^2+9}}{x^2}$.

7. a) $\lim\limits_{x\to -1}\frac{2x^2+x-1}{x^2-1}$; б) $\lim\limits_{x\to 2}\frac{\sqrt{2x-3}-1}{\sqrt{2x+3}-\sqrt{7}}$.

8. a) $\lim\limits_{x\to 5}\frac{2x^2-12x+10}{2x^2-11x+5}$; б) $\lim\limits_{x\to\infty}\frac{4x^8+5x-1}{3x^2+2x}$.

9. a) $\lim\limits_{x\to 4}\frac{\sqrt{x+5}-3}{\sqrt{x}-2}$; б) $\lim\limits_{x\to\infty}\frac{-7x+5-\sqrt{9x^2-7x}}{2x+1}$.

10. a) $\lim\limits_{x\to 3}\frac{\sqrt{7x^2-9x}-6}{x-3}$; б) $\lim\limits_{x\to -1}\frac{-4x^2-3x+1}{8x^3+2x^2-10x-4}$.

11. а) $\lim\limits_{x\to -1}\frac{-3x^2-3x}{-x^2+3x+4}$; б) $\lim\limits_{x\to\infty}\frac{9x-2+\sqrt[3]{4x^3+2x^2+2x-2}}{5x+1}$.

12. а) $\lim\limits_{x\to\infty}\left(9x-2-\sqrt{81x^2}\right)$; б) $\lim\limits_{x\to -1}\frac{\sqrt{4x^2-3x-3}-2}{x+1}$.

13. а) $\lim\limits_{x\to\infty}\left(8x+8-\sqrt{64x^2-8}\right)$; б) $\lim\limits_{x\to\infty}\frac{5x^4+3x^2-1}{7x^8+4}$.

14. а) $\lim\limits_{x\to\infty}\left(\sqrt{x-3}-\sqrt{x+2}\right)$; б) $\lim\limits_{x\to 3}\left(\frac{1}{x-3}-\frac{6}{x^2-9}\right)$.

15. а) $\lim\limits_{x\to 3}\frac{x^2-6x+9}{x^2-3x}$; б) $\lim\limits_{x\to 2}\frac{\sqrt{3x-2}-2}{\sqrt{x}-\sqrt{2}}$.

16. а) $\lim\limits_{n\to\infty}\frac{\left(3n^2+5\right)(n-1)!}{(n+1)!}$; б) $\lim\limits_{x\to\infty}\frac{\sqrt[4]{x^8+2x-10}-3x^2}{5x^2-1-\sqrt[3]{27x^6+x^5-15x}}$.

17. а) $\lim\limits_{x\to\infty}\frac{10x^4-3x^2+\sqrt[3]{x}}{\sqrt[5]{x}+7x^3-2x^4}$; б) $\lim\limits_{x\to\infty}\frac{3^x+2}{3^{x+1}-1}$.

18. а) $\lim\limits_{x\to\infty}\frac{(x-1)x!}{(x+1)!}$; б) $\lim\limits_{x\to 1}\frac{2x^2-14x+12}{x^2-6x+5}$.

19. а) $\lim\limits_{x\to\infty}\frac{(x+1)!-x!}{(x+1)!}$; б) $\lim\limits_{x\to 4}\frac{\sqrt{x+5}-3}{\sqrt{x}-2}$.

20. а) $\lim\limits_{x\to\infty}\frac{\sqrt{9x^2-9}-2x}{2-\sqrt[3]{x^3+5}}$; б) $\lim\limits_{x\to 0}\frac{2-\sqrt{x^2+4}}{3x^2}$.

II

1. а) $\lim\limits_{x\to\infty}x\left(\ln(2x-1)-\ln(2x-3)\right)$; б) $\lim\limits_{x\to 0}\frac{1-\cos 2x}{3x\sin x}$.

2. а) $\lim\limits_{x\to 0}\frac{\sin 3x+\sin x}{2x}$; б) $\lim\limits_{x\to 0}(1+2x)^{\frac{3+x}{x}}$.

3. а) $\lim\limits_{x\to\frac{\pi}{2}}\left(\frac{\pi}{2}-x\right)\cdot \mathrm{tg}x$; б) $\lim\limits_{x\to\infty}\left(\frac{3x}{1+2x}\right)^{-5x}$.

4. a) $\lim\limits_{x\to 2}\dfrac{-3x+6}{\sin(x-2)}$; б) $\lim\limits_{x\to 0}(1+4x)^{\frac{6}{x}+2}$.

5. a) $\lim\limits_{x\to\infty}\left(\dfrac{x}{x+3}\right)^{3x-2}$; б) $\lim\limits_{x\to 0}(1+2x)^{\frac{3+x}{x}}$.

6. a) $\lim\limits_{x\to\frac{\pi}{2}}(1+2\cos x)^{\frac{3}{\cos x}}$; б) $\lim\limits_{x\to\infty}\left(\dfrac{x=3}{x+4}\right)^{-2x}$.

7. a) $\lim\limits_{x\to\infty}[2x(\ln(x+3)-\ln(x-3))]$; б) $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{x^2}$.

8. a) $\lim\limits_{x\to 0}\dfrac{\mathrm{tg}x-\sin x}{x^3}$; б) $\lim\limits_{x\to 0}\left(\dfrac{4x^2-1}{3x^2-1}\right)^{\frac{3}{x^2}}$.

9. a) $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}}$; б) $\lim\limits_{x\to 0}\dfrac{\ln(5-x^2)-\ln 5}{2x^2}$.

10. a) $\lim\limits_{x\to 0}\left(\dfrac{3-2x^2}{3+3x^2}\right)^{-\frac{4}{x}}$; б) $\lim\limits_{x\to 0}\dfrac{7x}{\sin x+\sin 7x}$.

III. Find the limits using equivalent infinitesimal.

1. $\lim\limits_{x\to 0}\dfrac{\mathrm{arctg}2x}{8x}$

2. $\lim\limits_{x\to 0}\dfrac{\ln(1+5x)}{10x}$

3. $\lim\limits_{x\to 0}\dfrac{e^{5x}-1}{\sin 2x}$

4. $\lim\limits_{x\to -2}\dfrac{\mathrm{tg}(x+2)}{x^2-4}$

5. $\lim\limits_{x\to 0}\dfrac{\arcsin 2x}{\mathrm{tg}4x}$

6. $\lim\limits_{x\to 0}\dfrac{\mathrm{arctg}5x}{\mathrm{tg}2x}$

11. $\lim\limits_{x\to 0}\dfrac{\sin^2 3x}{\mathrm{arctg}^2 2x}$

12. $\lim\limits_{x\to 0}\dfrac{\mathrm{arctg}3x}{4x}$

13. $\lim\limits_{x\to 0}\dfrac{\sin 5x}{\arcsin 2x}$

14. $\lim\limits_{x\to 0}\dfrac{\arcsin^2 x}{3x\sin x}$

15. $\lim\limits_{x\to 0}\dfrac{5x}{\mathrm{arctg}3x}$

16. $\lim\limits_{x\to 0}\dfrac{\arcsin 6x}{2x}$

7. $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\ln(1+2x)}$

8. $\lim\limits_{x\to 0}\dfrac{e^{5x}-1}{\mathrm{tg}2x}$

9. $\lim\limits_{x\to 0}\dfrac{\arcsin 8x}{\mathrm{tg}4x}$

10. $\lim\limits_{x\to 0}\dfrac{\mathrm{arctg}6x}{2x^2-3x}$

17. $\lim\limits_{x\to 0}\dfrac{\sin 5x+\sin x}{\arcsin x}$

18. $\lim\limits_{x\to 0}\dfrac{1-\cos^2 2x}{x\arcsin x}$

19. $\lim\limits_{x\to 0}\dfrac{\arcsin 6x}{x^2-x}$

20. $\lim\limits_{x\to 0}\dfrac{4x^2-2x}{\mathrm{arctg}8x}$.

References

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